

ASSOCIATIVITY OF σ -SETS FOR NON-ANTIELEMENT σ -SET GROUPS

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ABSTRACT. We study and extend the conditions for asociativity on fusion over antielement free σ -sets to introduce a group to solve σ -set equations. σ -sets as a theory of sets and antisets is sumarized and used as a framework to define the main elements of this work. A theorem on Local Asociativity, conmutative groups and solution of one variable fusion equation is presented.

1. INTRODUCTION

The inverse element play a very important role, together with associativity, on solving equations form an algebraic structure. But what happend when the inverse element and associativity together arise unsolvable inconsistencies? Let us pose the following example. Let A and B two sets differet to empty, with the propriety that $A \cup B = \emptyset$. Associativity of a binary operation $*$ is defined as $(A*B)*C = A*(B*C)$, which in this context would be $(A \cup B) \cup C = A \cup (B \cup C)$. Let us say, finally, that $B = C$. So we have,

$$\begin{aligned}(A \cup B) \cup C &= \emptyset \cup C, \\ A \cup (B \cup C) &= A \cup B.\end{aligned}$$

As $\emptyset \cup C = C$, where, if asociativity holds under our previous assumptions, we would conclude that $B = \emptyset$, which contradict our first assumption.

That is the case of set theory, where this problem arised after the aparition of the concept of antisets. Antisets are a relativley new object studied on set theory, varing from elimination propriety under union [Car09], [Ar09], to gaps on a set which eliminate any element that join with [Bal13]; in this article, however, we will strictly use the first type of Antisets, particulary the definition developed by Arraus [Ar09].

What we want to present in this work, is the algebraic condition which aloud local asociativity of fussion on σ -sets, this is, extend the Arraus condition of asociativity on antielement free σ -sets and open the possiblity to associate sets and antisets. Our goal is to to make partition equations of the form

$$A \cup X = B,$$

as where presented on [Bus11], solvabe, for that matter, we will define a special type of group based on the locally asociative σ -set.

Before we present our results, we want to introduce some notation of Arraus work, which will help us to analize and define some of the concepts needed to proof our main theorem on asociativity on σ -sets.

2. PREELIMINARS ON σ -SETS

In [Ar09] it was defined a type of set denoted as σ -set, which was the gathering of both sets and antisets, where for a set A and a set A^* on a σ -set, the following identity holds,

$$A \cup A^* = \emptyset,$$

which is exactly replied on their elements, this is, for $X = \{x\}$ and $X^* = \{x^*\}$, $\{x, x^*\} = \emptyset$.

For X and Y σ -sets, he defined two base operations,

- (1) $X \hat{\cap} Y := \{x \in X : x^* \in Y\}$
- (2) $X * Y := X - X \hat{\cap} Y$.

Below, we paraphrase an example applying the above definitions.

Let $X = \{1, 2\}$ and $Y = \{1^*, 2^*\}$, where,

- (1) $X \hat{\cap} Y = X$
- (2) $X * Y = \emptyset$.

From this concepts, the author also defined what he denoted as **antielement free** σ -set, to a set X , where for $A, B \in X$, $A \hat{\cap} B = \emptyset$.

This σ -set was later denoted as,

$$AF = \{x : x \text{ is antielement free}\}.$$

A variation of union, called fusion, presented by Arraus at the beginning of the article, is redefined as,

$$X \cup Y = \{x : (x \in X * Y) \vee (x \in Y * X)\}$$

Finally, a theorem is introduced in order to show why associativity is not possible in fusion over σ -sets, this is, on not antielement free σ -sets.

Theorem 2.1 (Th.3.52, p.26). *Let X be a σ -set. Then*

- (a): $X \cup \emptyset = \emptyset \cup X = X$
- (b): $X \cup X = X$.

It is shown that fusion is commutative, this is $X \cup Y = Y \cup X$, but when associativity is to be shown, the author point out the following case. Let us have the σ -sets $X = \{1, 2\}$, $Y = \{1^*, 2^*\}$ and $Z = \{1\}$, where,

$$(X \cup Y) \cup Z = \emptyset \cup Z = Z$$

and

$$X \cup (Y \cup Z) = X \cup Y = \emptyset,$$

where, as in the example presented in this article, occurs the same contradiction where $Z = \emptyset$.

To solve this problem, the author proposed the following theorem,

Theorem 2.2. *If $X \in AF$, then the fusion in X is associative, that is*

$$(X \in AF)[(A \cup B) \cup C = A \cup (B \cup C)].$$

Under this scheme, the σ -sets, to have the propriety of being associative, there must be antielement free. But what happens when equations like $A \cup X = B$ has to be solved? Is there a restriction from which associativity on σ -sets holds? Such restriction is our main result, a theorem on the condition of associativity on σ -sets, which we will present in the following section together with some definitions and lemmas to prove it.

3. DEFINITIONS AND MAIN RESULT

3.1. Associative chain of fusion and Locally associative σ -Sets. Before introduce our main result, let us present the following previous definitions and lemmas.

We are going to take the notion of **chain of fusion** which was defined by Arraus as,

$$\bigcup \vec{\mathcal{F}} = (A \cup B) \cup C,$$

which for the prupose of the following definitions, we will denote as \overrightarrow{ABC} , where ABC represent a fixed order on the expression.

Definition 3.1. Let A, B and C σ -sets on S . An **Associativity Evaluation Chain** on S , denoted as E_S , is an equation on σ -sets, where,

$$E_S = \overrightarrow{ABC} \cup \overrightarrow{C^*B^*A^*}.$$

Definition 3.2. An **Associative chain of fusion over S** is an asociative chain where, for every A, B and C σ -sets on S ,

$$(A \cup B) \cup C = A \cup (B \cup C).$$

Lemma 3.3. Let A, B and C be σ -sets of S . if $(A \cup B) \cup C = A \cup (B \cup C)$, then $\overrightarrow{ABC} = \overrightarrow{CBA}$.

Proof. Is enough to prove that, by commutativity, $A \cup (B \cup C) = (C \cup B) \cup A$ because $X = A$ and $Y = B \cup C$; the same with $B \cup C = C \cup B$, where we have the fusion chain \overrightarrow{CBA} . As we previously assumed that $(A \cup B) \cup C = A \cup (B \cup C)$, and we know by definition that $\overrightarrow{ABC} = (A \cup B) \cup C$, proving the requested identity. \square

Lemma 3.4. If \overrightarrow{ABC} is a σ -set, then $\overrightarrow{A^*B^*C^*}$ is it antiset.

Proof. By definition of antiset and chain of fusion over $\overrightarrow{A^*B^*C^*}$, is enough to show that the fusion of \overrightarrow{ABC} and $\overrightarrow{A^*B^*C^*}$ is actually the empty set. \square

Proposition 3.5. Let A, B and C be σ -sets on S . if $E_S = \emptyset$, then fusion over S is associative.

Proof. Let us show by contapositive. Assume that $(A \cup B) \cup C \neq A \cup (B \cup C)$. We know that the second expression by the law of commutativity on fusion, we have fusion chain \overrightarrow{ABC} . If we fusion $(A \cup B) \cup C$, this is, the fusion chain \overrightarrow{ABC} with the antielement σ -set $\overrightarrow{C^*B^*A^*}$, we will obtain,

$$\overrightarrow{ABC} \cup \overrightarrow{C^*B^*A^*} \neq \emptyset,$$

because of $\overrightarrow{ABC} \neq \overrightarrow{CBA}$ by our hypothesis, hence the expression obtained is equivalent, by definition, to the equation E_S , so the proof is complete. \square

Let us illustrate the above proposition with the following example.

We have the σ -sets $A = \{a, b\}$, $B = \{a^*, b^*\}$ and $C = \{c, d\}$. Taking the definition of fusion, we have that

$$\overrightarrow{ABC} = C$$

and

$$\overrightarrow{C^*B^*A^*} = C^*,$$

where finally, the Associative evaluation chain $E_S = \emptyset$, so by the proposition, ABC is an associative chain of fusion.

Now let us take a contrary example to see what happend when the Associative evaluation chain is not \emptyset . We have the σ -sets $A = \{1, 2\}$, $B = \{1^*, 2^*\}$ and $Z = \{1^*\}$. Taking the definition of fusion, we have that

$$\overrightarrow{ABC} = Z,$$

and

$$\overrightarrow{C^*B^*A^*} = \emptyset,$$

where

$$E_S = Z \cup \emptyset = Z,$$

which is different to \emptyset , so we can conclude that ABC is not associative.

This results as we can see, is a very useful tool to evaluate associativity over an ordered configuration of σ -sets; however, we are still not able to evaluate on a wide combinatorial range, this is, over the generalized union $\bigcup\{A, B, C\}$. For thar reason, we develop the following definitions.

Definition 3.6. *Let A, B and C σ -sets. A, B, C are said to be **Locally Associative** over fusion if, for any combination of A, B and C , the fusion chain is associative.*

There exist some σ -sets S such that are associative chain of fusion but are not locally associative. Let us examine the following σ -set where $A = \{1, 2\}$, $B = \{1^*, 2^*\}$ and $C = \{1, 2\}$. As we can see, by definition of associative evaluation chain, we have that $E_S = \emptyset$, so is associative; but if we re-arrange the set on the fusion set as BAC , we will have that the new evaluation chain would be diferent to empty set, implyng that the associativity is not defined for every configuration.

Definition 3.7. *For the equation system $E = \{E_X, E_Y, E_Z\}$, where the σ -sets X, Y, Z are in E , the equations*

$$E_X = \overrightarrow{XYZ} \cup \overrightarrow{Z^*Y^*X^*},$$

$$E_Y = \overrightarrow{YZX} \cup \overrightarrow{X^*Z^*Y^*},$$

$$E_Z = \overrightarrow{ZXY} \cup \overrightarrow{Y^*X^*Z^*}.$$

Theorem 3.8. *For X, Y and Z σ -sets, if $E = \{\emptyset\}$, then fusion is locally associative on σ -sets.*

Proof. As $E = \{\emptyset\}$ we know that the equations E_X, E_Y and E_Z are equal to the empty set, wich implies that XYZ, YZX and ZXY are commutative fusion chains. Because of lemma 3.3, same ZYX, XZY and YXZ are. As every configuration is an associative fusion chain, by definition, is locally associative. \square

3.2. Main Theorem on σ -sets Conmutative groups. To start this section, let us remind what is a group as an algebraic structure. First, we remember that an algebraic structure [Bo70] is a pair $(A, *)$ where A is a set of certain mathematical objects, and $*$ is an action from $A \times A$ to A , this is, the result of the operation of two elements of A also belong to A .

Algebraic structures posess also composition laws, this is, laws of what happend when operations do certain combination of special objects of the set or coclatenations like associativity.

A group $(G, *)$ is a set G of elements a, b, c such that,

- (G1) $a * e = a$
- (G2) $a * a^{-1} = e$
- (G3) $(a * b) * c = (a * b) * c,$

if in the group $a * b = b * a$, it said to be conmutative or abelian. The group we are going to study, is an abelian one, we will proof that the theorem 3.8. allows the minumal conditions to define a cancelative group.

Theorem 3.9. *Let A, B, C be σ -sets on \mathbb{G} . if \mathbb{G} is locally associative, then (\mathbb{G}, \cup) is an associative group.*

Proof. As \mathbb{G} is locally associative, means that $E = \{\emptyset\}$ so, there is neutral element; in the other hand, as every E_X, E_Y, E_Z in E has in their expression an antiset, where by definition of evaluation chain and lemma 3.2, the rest of the needed condition to be a group are acomplished. \square

Theorem 3.10. *If (S, \cup) is a Locally associative group, then $X \cup A = B$ is solvable and the solution is $X = B \cup A^*$.*

Proof. the σ -set A, B, C is locally asociative, so, if we left add to both sides of the equation the term A^* , the right expresion $(X \cup A) \cup A^*$, by local associativity and cancellation with antisets, you will have X , the left term remains without change so we can conclude that has a solution. \square

To have a better understandig of this result, let us ilustrate it with the following example.

Let us have a σ -set S where $A = \{\alpha, \beta\}$, and $B = \{a^*, b^*, c^*, \alpha, \beta\}$, and X is to be found in the form $A \cup X = B$. First, we separate the expression

$$\{a^*, b^*, c^*, \alpha, \beta\} = \{a^*, b^*, c^*\} \cup \{\alpha, \beta\}$$

then, by definition of locally associative σ -set, we know that X is part of an associative chain of fusion over any permutation. This aloud us to just aply the theorem on result for this kind equations, so we have,

$$X = (\{a^*, b^*, c^*\} \cup \{\alpha, \beta\}) \cup \{\alpha^*, \beta^*\},$$

because this is an associative chain, is enough to apply associativity and cancelation over σ -sets to finaly obtain that $X = \{a^*, b^*, c^*\}$ which is the solution to obtain the sigma set B .

4. FURTHER APPLICATIONS

A possible application of this research is on link algebra, where by the theorem on locally associative σ -sets over fusion, it is possible to make graph equations over union, as equations of the form $A \cup X = B$, as where presented on [Bus11], solvable.

There is an other case where this kind of special σ -set space could be applied: graph join equations, where a graph G_1 is joined to a graph G_2 if all the vertices of the two graphs are joined by edges. This type of result, could constitute a case of a join graph equation of one variable, where the inverse graph G_2^* could provide the graph to obtain the resultant graph, this would be specially useful on Zykov's Linear Complexes [Zy49], where graph join was denoted as $G_1 G_2$, in order to solve problems on graph concentration.

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